

## *Invited Review*

# From NBODY1 to NBODY6: The Growth of an Industry<sup>1</sup>

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**ABSTRACT.** I review the development of direct  $N$ -body codes at Cambridge over nearly 40 years, highlighting the main stepping stones. The first code (NBODY1) was based on the simple concepts of a force polynomial combined with individual time steps, where numerical problems due to close encounters were avoided by a softened potential. Fortuitously, the elegant Kustaanheimo-Stiefel two-body regularization soon permitted small star clusters to be studied (NBODY3). Subsequent extensions to unperturbed three-body and four-body regularization proved beneficial in dealing with multiple interactions. Investigations of larger systems became possible with the Ahmad-Cohen neighbor scheme which was used more than 20 years ago for expanding universe models of 4000 galaxies (NBODY2). Combining the neighbor scheme with the regularization procedures enabled more realistic star clusters to be considered (NBODY5). After a period of simulations with no apparent technical progress, chain regularization replaced the treatment of compact subsystems (NBODY3, NBODY5). More recently, the Hermite integration method provided a major advance and has been implemented on the special-purpose HARP computers (NBODY4) together with an alternative version for workstations and supercomputers (NBODY6). These codes also include a variety of algorithms for stellar evolution based on fast lookup functions. The treatment of primordial binaries contains efficient procedures for chaotic two-body motion as well as tidal circularization, and special attention is paid to hierarchical systems and their stability. This family of  $N$ -body codes constitutes a powerful tool for dynamical simulations which is freely available to the astronomical community, and the massive effort owes much to collaborators.

## 1. INTRODUCTION

The title of this talk was conceived at 20,000 feet last year on my rest day during a solo climb on Lullailaco in the Atacama Desert. It was inspired by an encounter with V. A. Ambartsumian more than 20 years ago over Armenian cognac when he told me that I had started a new industry. The popularity of  $N$ -body codes has increased considerably since then, and given my continued interest and the subsequent activity in this subject, his prophetic utterance might seem justified.

Broadly speaking, my association with the  $N$ -body problem can be connected with the development of the six main codes alluded to in the title, which happen to have been given mundane names, rather than some clever acronym so prevalent in recent years. The codes listed below in Table 1 also represent an evolutionary sequence, although the actual numbering is historical and intended to suggest the relationship of the relevant methods, and these codes are mostly in order of increasing sophistication.

Because these codes range in size from 2000 to about 34,000 lines (including comments), it will be appreciated that life has become complicated for the modern practitioner. Still, the guiding light for the  $N$ -body code builder must be clarity and elegance, which requires endless attention in order to create a satisfactory final product.

These basic codes were developed in response to various algorithms which became available, mostly through the work of others. Because my scientific life has to a large extent been influenced by the challenge of converting new ideas into useful numerical procedures, it may be appropriate to mention the most important ones. A number of so-called algorithmic stepping stones are summarized in Table 2, with the year of implementation in the first column. The second column contains an appropriate keyword, while the last column points to the corresponding reference. From an inspection of this list it becomes apparent that progress was slow to begin with. This development is a reflection of the hardware situation in the early days as well as the low-level activity regarding both collaborators and general scientific exchange through meetings. Only an old-timer can appreciate the vast difference between keeping in touch by letters

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<sup>1</sup> Brouwer Award Lecture.

TABLE 1  
MAIN  $N$ -BODY CODES

Keyword	Period	Name
Primitive beginnings .....	1961–1969	NBODY1
Two-body regularization .....	1969–1974	NBODY3
Cosmological experiments .....	1974–1983	NBODY2
Star cluster simulations .....	1979–1992	NBODY5
Hermite integration .....	1993–1999	NBODY6
The HARP challenge .....	1994–1999	NBODY4

(two months response for the Soviet Union) and the instant communication by e-mail taken for granted today.

In the following sections the topics listed in Table 2 will be described, highlighting the most important in some detail.

## 2. FORCE POLYNOMIAL

In order to get started on the  $N$ -body problem, it is essential to find a good strategy for dealing with the expensive force summation. In retrospect, it was a stroke of luck that a polynomial fitting function was decided upon from the very beginning, albeit at low order (Aarseth 1963). Later the order was increased from three to four and a more elegant divided difference formulation was adopted (Ahmad & Cohen 1973).

TABLE 2  
ALGORITHMIC STEPPING STONES

Year	Keyword	Reference
1961.....	Force polynomial	Aarseth 1963
	Individual time steps	Aarseth 1963
	Softening	Aarseth 1963
1966.....	Spherical harmonics	Aarseth 1967
1969.....	Two-body regularization	Kustaanheimo & Stiefel 1965
1972.....	Three-body regularization	Aarseth & Zare 1974
1973.....	Global regularization	Heggie 1974
	Neighbor scheme	Ahmad & Cohen 1973
1978.....	Comoving coordinates	Aarseth 1979
1979.....	Regularized AC	Aarseth 1985
1980.....	Planetary formation	Lecar & Aarseth 1986
1989.....	Chain regularization	Mikkola & Aarseth 1990
1990.....	Particle in box scheme	Aarseth, Lin, & Palmer 1993
1991.....	Collisional tree code	McMillan & Aarseth 1993
1992.....	Chain $N$ -body interface	Aarseth 1994
1993.....	Hermite integration	Makino 1991
1995.....	Synthetic stellar evolution	Tout et al. 1997
	Tidal circularization	Mardling 1995
	Slow chain regularization	Mikkola & Aarseth 1996
1996.....	Hierarchical stability	Mardling & Aarseth 1999a
1998.....	Evolution of hierarchies	Mardling & Eggleton 1999
	General stellar evolution	Hurley & Tout 1999
	Stumpff KS method	Mikkola & Aarseth 1998
1999.....	HARP-6 procedures	Aarseth 1999b

For each particle we write the fourth-order fitting function evaluated at the times  $t_k$  in the form

$$F_t = (\{[D^4(t - t_3) + D^3](t - t_2) + D^2\}(t - t_1) + D^1) \times (t - t_0) + F_0, \quad (1)$$

where the divided differences are defined by

$$D^k[t_0, t_k] = \frac{D^{k-1}[t_0, t_{k-1}] - D^{k-1}[t_1, t_k]}{t_0 - t_k}; \quad (k = 1, 2, 3). \quad (2)$$

Differentiating this expression and setting  $t_k = t$  gives rise to the Taylor series relations

$$F^{(1)} = [(D^4 t'_3 + D^3)t'_2 + D^2]t'_1 + D^1, \quad (3)$$

$$F^{(2)} = 2![D^4(t'_1 t'_2 + t'_2 t'_3 + t'_1 t'_3) + D^3(t'_1 + t'_2) + D^2], \quad (4)$$

$$F^{(3)} = 3![D^4(t'_1 + t'_2 + t'_3) + D^3], \quad (5)$$

$$F^{(4)} = 4!D^4, \quad (6)$$

with  $t'_k = t_0 - t_k$ . The Taylor series expansion is used for the integration, whereas the divided differences and corresponding times are saved in memory. Moreover, the contributions from the fourth derivative to *all* the lower derivatives are added at the end as a so-called semi-iteration, which plays the role of a corrector. This device was actually introduced by von Hoerner (1960) in his pioneering paper.

It was also realized early (Aarseth 1963) that force polynomials facilitate prediction of coordinates so that individual time steps may be chosen. A simple form of the time step is given by a relative criterion of the type

$$\Delta t_i = \left( \frac{\eta |F|}{|F^{(2)}|} \right)^{1/2}, \quad (7)$$

where  $\eta$  is a dimensionless constant. In most situations, this expression ensures convergence of the force polynomial; however, a more sensitive criterion involving all the force derivatives was adopted subsequently (Aarseth 1985). The individual time step scheme requires new coordinates of all other particles to be predicted at each force evaluation. Fortunately, only a low-order prediction is called for here which enables a fast evaluation, and hence the additional cost is relatively modest; in return, a wide range of time steps may be used. For completeness, we write the coordinate prediction in the form

$$r_j = \left[ \left( \frac{1}{6} F^{(1)} \delta t'_j + \frac{1}{2} F \right) \delta t'_j + v_0 \right] \delta t'_j + r_0, \quad (8)$$

with  $\delta t'_j = t - t_j$ . In the individual time step scheme the

current time  $t$  is defined as  $t = t_i + \Delta t_i$ , where particle  $i$  is determined formally from minimizing all such expressions, although a faster algorithm is used in practice. We note that such a scheme requires the additional variables  $v_0$ ,  $r_0$  evaluated at the beginning of each step.

The third entry in Table 2 refers to a softening of the Newtonian potential which was introduced in order to model galaxy clusters. It turns out that the adopted expression

$$\Phi = \frac{Gm}{(r^2 + \epsilon^2)^{1/2}} \quad (9)$$

represents a Plummer sphere, with half-mass radius related to the softening size by  $r_h \simeq 1.3\epsilon$  (see Aarseth & Fall 1980). In many subsequent simulations of collisionless systems a softening size  $\epsilon \simeq r_h/N$  was chosen in order to reduce two-body relaxation and mass segregation. In retrospect, this modification avoided numerical problems associated with close encounters and persistent binaries which had to wait for new techniques to become available, and hence the early experience served as an easy introduction to direct  $N$ -body simulations. Still, it proved possible to study the long-term behavior of hard binaries in a star cluster model with  $N = 250$  members using a relatively small value of the softening parameter (Aarseth 1968).

On a historical note, when this work was started in 1961 at the suggestion of F. Hoyle, we were not aware of the paper by von Hoerner (1960), and our effort was therefore independent as can be seen from the different algorithms. The original version of NBODY1 (Aarseth 1963) was based on a third-order Taylor series for the force polynomial with backward differences. It was later extended to fourth order (Aarseth 1968) and eventually converted to divided differences introduced by Ahmad & Cohen (1973), as adopted in the subsequent codes. Amazingly, NBODY1 has survived and is sometimes used for educational purposes (Binney & Tremaine 1987) and even scientific work (McDonald & Clarke 1995; Hjorteland 1999). It is currently maintained on the World Wide Web as part of the NEMO software package<sup>2</sup> (Teuben 1995).

### 3. KS REGULARIZATION

The realization of the elegant Kustaanheimo-Stiefel (1965, hereafter KS) two-body regularization provided a much needed stimulus for those interested in the classical  $N$ -body problem. Its versatility was demonstrated in the entertaining three-body study of Szebehely & Peters (1967) and by the time of the Cambridge IAU Colloquium in 1970

had found an application in general  $N$ -body codes (Aarseth 1972b; Bettis & Szebehely 1972). This beautiful theory is the outcome of many efforts toward such a goal during 60 years since the two-dimensional Levi-Civita regularization. The main stumbling block was the impossibility of generalizing the complex plane representation to a three-dimensional mapping, and the final success was only possible by introducing a four-dimensional transformation with some redundancy. Although the KS formulation was very timely and had a profound influence on my own work, the number of practitioners has been small in spite of favorable publicity.

The basic idea of KS regularization is to introduce a set of four new coordinates  $\mathbf{u}_k$  satisfying the relation

$$R = u_1^2 + u_2^2 + u_3^2 + u_4^2, \quad (10)$$

where  $R$  is the physical separation. This is combined with the time transformation

$$dt = R d\tau \quad (11)$$

which connects physical and regularized time. The achievement of KS was to provide a coordinate transformation which satisfies the basic relation equation (10). Thus the physical coordinates are obtained by the transformation

$$\mathbf{R} = \mathcal{L}(\mathbf{u})\mathbf{u}, \quad (12)$$

where the linear Levi-Civita matrix is given by

$$\mathcal{L}(\mathbf{u}) = \begin{bmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{bmatrix}. \quad (13)$$

Because of the redundancy, it is necessary to define a fourth component of  $\mathbf{R}$  and its derivative to be zero. Likewise, the relative velocity is evaluated from

$$\dot{\mathbf{R}} = \frac{2\mathcal{L}\mathbf{u}'}{R}, \quad (14)$$

where differentiation with respect to the fictitious time is denoted by primes. The resulting equations of motion take the form

$$\mathbf{u}'' = \frac{1}{2}h\mathbf{u} + \frac{1}{2}R\mathcal{L}^T\mathbf{P}, \quad (15)$$

$$h' = 2\mathbf{u}' \cdot \mathcal{L}^T\mathbf{P}, \quad (16)$$

$$t' = \mathbf{u} \cdot \mathbf{u}, \quad (17)$$

where  $h$  is the binding energy per unit reduced mass and  $\mathbf{P}$  is

<sup>2</sup> <http://www.astro.umd.edu/nemo>.

the perturbation expressed in physical variables.

For a complete description of the motion, the center of mass is introduced as a fictitious particle, with its own appropriate force polynomial. Consequently, the space motion of a KS pair is obtained by integrating the center of mass as a single particle together with the relative motion. Although there are overheads connected with transformations, it is generally more efficient to adopt KS regularization for dominant two-body motion rather than integrate the two particles by separate polynomials.

The main decision making in the code NBODY3 is based on two parameters. Candidates for new KS treatment are selected by a time step criterion,  $\Delta t_i < \Delta t_{ci}$ , subject to  $R < r_{ci}$ , where the close encounter distance  $r_{ci} \simeq r_h/N$  and  $\Delta t_{ci}$  is the corresponding time step for a typical hyperbolic encounter estimated from the integration scheme. Termination of a regularization is based on the relative perturbation  $\gamma \equiv |\mathbf{P}|R^2/m_b$ , where  $m_b$  is the mass of the binary. Hard binaries are generally terminated for  $\gamma > 0.5$ , whereas soft binaries and hyperbolic orbits are replaced by direct integration at somewhat smaller values.

Because the tidal force on the two-body motion falls off as  $1/r^3$ , only the neighbors inside  $\lambda R$  are included as perturbers, where the choice  $\lambda \simeq 100$  implies a limiting perturbation  $\gamma \simeq 1 \times 10^{-6}$  for equal-mass particles. Consequently, we adopt unperturbed motion if there are no perturbers at the apocenter turning point. Depending on the velocity field, it is often possible to extend this interval to a number of Kepler orbits, thereby reducing the cost of binary integration to that of single particles.

To conclude on an anecdotal point, the existence of KS was first brought to my attention at IAU Symposium 25 in 1964, which took place well *before* the actual publication of the method. The Proceedings of IAU Symposium 25 are in fact dedicated to the memory of Dirk Brouwer. On this special occasion (also my first conference), V. Szebehely asked me the leading (abbreviated) question, "Why does the speaker deny himself the pleasure offered by regularization," which drew a blank response, subsequently modified for the record (Aarseth 1966). Later it emerged that E. Stiefel was a regular visitor to Szebehely's department at Austin.

#### 4. THREE-BODY REGULARIZATION

Star cluster simulations inevitably feature strong interactions between hard binaries and single stars or even other binaries which may pose difficulties for the standard KS treatment. The switching of dominant components reduces the efficiency and also degrades the accuracy. By a remarkable piece of good luck, a powerful method for dealing with close triple encounters was developed as early as 1972 when the author visited the University of Texas, Austin, and com-

bined efforts with K. Zare, who exploited the separable generating functions suggested by Szebehely (1967).

The basic idea is to introduce two coupled KS regularizations with coordinate transformations for the distances  $R_1, R_2$  from  $m_1$  and  $m_2$  to the reference body  $m_3$  by

$$\mathbf{Q}_k^2 = R_k ; \quad (k = 1, 2), \quad (18)$$

together with the time transformation

$$dt = R_1 R_2 d\tau . \quad (19)$$

As the first step we reduce the order of the 18 differential equations to 12 by employing the six integrals of the center-of-mass motion. Because the corresponding equations of motion describe a system with six independent coordinates, it is natural to perform regularization in eight dimensions by increasing the set of physical variables in analogy with the KS formulation. Combining separable generating functions with the standard Hamiltonian development in the extended phase space (Szebehely 1967) and multiplying by  $R_1 R_2$ , we obtain the regularized Hamiltonian function

$$\begin{aligned} \Gamma^* = & \sum_{k=1}^2 \frac{1}{8\mu_{k3}} R_l \mathbf{P}_k^2 + \frac{1}{16m_3} \mathbf{P}_1^T \mathbf{A}_1 \cdot \mathbf{A}_2 \mathbf{P}_2 - m_1 m_3 R_2 \\ & - m_2 m_3 R_1 - \frac{m_1 m_2 R_1 R_2}{|\mathbf{R}_1 - \mathbf{R}_2|} - E_0 R_1 R_2 . \end{aligned} \quad (20)$$

Here  $l$  is the integer part of  $(k+1)/k$  and  $\mu_{k3} = m_k m_3 / (m_k + m_3)$  is the reduced mass of each two-body motion. The second term contains products of the regularized momentum  $\mathbf{P}_k$  and transformation matrix  $\mathbf{A}_k$  of the interaction between  $m_1$  and  $m_2$ . We now require that the numerical value of the Hamiltonian function stay close to the initial total energy  $E_0$  on the solution path. The corresponding expressions for the equations of motion are given by

$$\frac{d\mathbf{Q}_k}{d\tau} = \frac{\partial \Gamma^*}{\partial \mathbf{P}_k} ; \quad \frac{d\mathbf{P}_k}{d\tau} = -\frac{\partial \Gamma^*}{\partial \mathbf{Q}_k} . \quad (21)$$

These equations are regular for  $R_1 \rightarrow 0$  or  $R_2 \rightarrow 0$  since  $\mathbf{P}_k \rightarrow \text{const}$  as  $R_k \rightarrow 0$ , and hence a practical regularization of the three-body problem has been achieved (Aarseth & Zare 1974).

Examination of the singular interaction terms shows that after differentiation these terms are *numerically* smaller than the regular contributions, provided  $|\mathbf{R}_1 - \mathbf{R}_2| > \max(R_1, R_2)$ . This useful property is mainly responsible for the improved treatment of close triple encounters, which can now be studied at high accuracy and reduced cost. Although some switching of the reference body may still occur, the same configuration can be maintained until the

distance between  $m_1$  and  $m_2$  becomes the *smallest*. Hence it can be seen that the decision making is extremely simple, which is an attractive feature of this scheme. Finally, we note that although quite large mass ratios may be studied, the presence of the reduced mass in the denominator of equation (20) prevents an application to the restricted three-body problem.

The idea of two coupled KS transformations was extended by Heggie (1974), who generalized this formulation to a *global* regularization of all the  $N(N-1)/2$  interactions. In the case of  $N=3$ , the number of equations increases from 17 to 25. In spite of its elegance, the global method is only superior for motion near the triple collision singularity, which remains undetermined. However, the corresponding reformulation for  $N=4$  (Mikkola 1985) was used for some time (until superseded by chain regularization) in order to treat binary-binary collisions in star cluster simulations.

## 5. PLANETESIMAL DYNAMICS

The availability of a general  $N$ -body code leads naturally to the question of applications. Following earlier simulations based on semianalytical expressions which only included close encounter effects between massive planetesimals in the terrestrial zone (Cox & Lewis 1980), it seemed desirable to check the conclusion of incomplete accretion by a more consistent calculation. However, the inclusion of distant interactions between lunar-sized bodies is not essential during most of the evolution. A special method based on a two-dimensional perturber grid in azimuth and radius was therefore developed (Aarseth 1985) which included contributions out to some 60 Hill radii. Early results (Lecar & Aarseth 1986) demonstrated for the first time that a two-dimensional system of 200 planetesimals in the terrestrial zone can accumulate into six protoplanets on a timescale of only  $\simeq 3 \times 10^4$  yr. Further integration beyond  $1 \times 10^5$  yr did not reduce the number because of resonance effects in spite of two intersecting orbits. Although encouraging, these results still suffer the defect of too large final eccentricities which seem to require a model with an initial mass function (IMF) combined with dissipative processes, as well as a full three-dimensional simulation.

An improved model, albeit still two-dimensional, included fragmentation as well as more careful treatment of distant encounters (Beaugé & Aarseth 1990). Now the final configurations after  $\simeq 5 \times 10^5$  yr reduced to four main bodies with moderate eccentricities and appeared to be stable. However, these early investigations should be considered exploratory and reflect the limitations of computing power in the 1980s.

The approach of investigating an earlier stage by studying a self-similar system of low-mass planetesimals has also

been tried (Aarseth, Lin, & Palmer 1993). In this so-called particle-in-box method, the perturbations from eight surrounding boxes projected in two dimensions are included at considerable cost, where the interior and exterior boxes are subject to differential rotation. However, this formulation does allow flattened three-dimensional systems to be studied. With a box size of a few percent of the central distance and  $N \simeq 100$ , the characteristic mass of  $\simeq 1 \times 10^{-10} m_\odot$  represents a stage when gravitational interactions begin to modify the orbital parameters. The results show evidence for energy equipartition and some indication of early runaway coagulation, with an estimated growth time for protoplanetary cores at 1 AU of  $\simeq 1 \times 10^5$  yr. This technique can also be extended to include gas drag which may play an important role in the early stage.

## 6. COSMOLOGICAL INTERLUDE

Because of my early interest in galaxy clusters, it might seem a small step to go to the scale of superclusters. However, in order to describe real superclusters it becomes necessary to consider much larger memberships, for which the standard one-polynomial method becomes prohibitively expensive. The development of a neighbor scheme (Ahmad & Cohen 1973; hereafter AC) was very timely and provided an impetus to the study of increased particle numbers. Somewhat surprisingly, the AC scheme becomes more efficient than one force polynomial already at  $N \simeq 50$ , with a predicted gain of  $(N/3.8)^{1/4}$  for large memberships (Makino & Aarseth 1992).

The resulting NBODY2 code proved to be the workhorse for a number of collisionless investigations from the earliest days (e.g., White 1976), although it was not presented until much later (Aarseth 1985), with a complete description still due to be published (Aarseth 1999b). Another interesting early study concerned the violent relaxation of galaxies and galaxy clusters with aspherical initial conditions (Aarseth & Binney 1978). These simulations were the precursor of much subsequent work on triaxial dynamics.

Applications to cosmological problems were inspired by a long-term visit of J. R. Gott III to the Institute at Cambridge. The growth of galaxy clustering was studied in a series of papers (Aarseth 1978; Aarseth, Gott, & Turner 1979). Both open and closed models were investigated for up to 4000 galaxies, which were represented as mass points in order to increase the resolution. Particular emphasis was placed on the analysis of group membership and the covariance function (Gott, Turner, & Aarseth 1979). Some models were also examined to obtain information on the void distribution which can be used to test galaxy clustering theories (Aarseth & Saslaw 1982). Another galaxy clustering simulation (Aarseth & Fall 1980) illustrated the propensity for galaxies to merge in the absence of dark matter.

The involvement with expanding universe simulations led to the development of a more efficient AC code, COMOVE, based on comoving coordinates (Aarseth 1979). In this formulation, only the deviations from the Hubble flow are integrated; hence such a method is particularly suited to the study of open models. Although modest by recent standards, application to a system of  $10^4$  galaxies with pronounced peculiar velocities proved feasible on a VAX 11/780 (Dekel & Aarseth 1984).

To end this section on a historical note, it may be remarked that the numerical experiments in cosmology took place during the same period as the planetary investigations, which also bracketed a more realistic approach to star cluster modeling.

## 7. STAR CLUSTER SIMULATIONS

The earliest  $N$ -body studies of point-mass systems paid scant attention to the astrophysical aspects of real clusters, but eventually the galactic tidal field was added (Hayli 1970). The spatial extent of open clusters is governed by the tidal radius,

$$R_t = \left[ \frac{GM}{4A(A-B)} \right]^{1/3}, \quad (22)$$

where  $A$  and  $B$  are the Oort's constants of galactic rotation and  $M$  is the cluster mass. Subsequent simulations also included simple supernova-type mass loss from massive stars (Wielen 1972; Aarseth 1973). Open cluster models with 250 initial stars indicated a half-life of  $2.5 \times 10^8$  yr when using an escape criterion of  $2R_t$ , which is still the favored prescription. Hence stars outside this distance are unlikely to be counted as cluster members and in any case will tend to disperse rapidly, whereas stars near  $R_t$  may linger. The question of the predicted tidal flattening (Aarseth 1973; Wielen 1975) was eventually confirmed observationally (Oort 1979).

Binaries play an important role in the dynamical evolution of both small and large clusters. This feature was first examined systematically for small ( $N \leq 24$ ) systems (van Albada 1968), where the tendency for a massive binary to end up with at least the total cluster energy was noted. Extension to  $N = 500$  confirmed this general behavior (Aarseth 1972a) and was in fact the first significant simulation which employed the KS method. Moreover, these numerical results posed a challenge to theoreticians, which inspired new work providing an increased understanding of three-body processes (Heggie 1975).

Several investigations with up to 500 members still made use of the single force polynomial method. However, the availability of the code NBODY2 for the AC neighbor scheme prompted development of a regularized version by 1979, which was eventually renamed NBODY5 and

described somewhat later (Aarseth 1985). The AC scheme is particularly suited to studying binaries because the neighbor list can be used for determining the perturbers, which would otherwise need to be obtained by a full  $N$  search (as in NBODY3). In addition to standard KS regularization, unperturbed three-body and four-body regularization for compact subsystems was incorporated at later stages.

One of the main early applications of NBODY5 was a simulation of interstellar cloud encounters acting on open clusters (Terlevich 1987). Here the idea was to add the perturbing effect of five massive clouds in rectilinear orbits inside a radius of 30 pc, whose initial velocity at the boundary was selected from an appropriate Gaussian distribution. Somewhat surprisingly, the total disruption time was not significantly affected for clusters with up to 1000 stars, where lifetimes of at least  $5 \times 10^8$  yr were obtained. This work confirmed an earlier result for smaller clusters (Aarseth & Wolf 1972) that the preferential escape rate of low-mass stars appears to be relatively modest, in contrast with theoretical expectations and observational interpretations. Subsequent modeling of more realistic initial conditions, albeit without massive stars subject to stellar evolution effects, gave further support to the idea that the open cluster IMF is deficient in low-mass stars (Kroupa 1995b).

Later on, the first dedicated workstation at the Institute was used in a systematic investigation of the dynamical effects of primordial binaries in systems with 2500 members (Heggie & Aarseth 1992), where a typical model required some 2000 CPU hours. A more powerful workstation was kept busy for half a year and enabled  $N = 6000$  single particles and 180 binaries to be studied over nearly 1000 crossing times (Aarseth & Heggie 1993). These models, and others (McMillan, Hut, & Makino 1990), showed a larger core mass than similar systems without primordial binaries and demonstrated that core collapse is halted by binaries. However, the question of relative depletion of binaries could not be settled in the absence of mass loss. Much valuable data were accumulated subsequently but remained unpublished, overtaken by events (i.e., the HARP special-purpose computer). Also noteworthy was a heroic effort of integrating an equal-mass system of  $N = 10,000$  particles well into the postcollapse regime on a Cray YMP vector supercomputer (Spurzem & Aarseth 1996).

The NBODY5 code achieved considerable popularity among astronomers and was without a rival for nearly two decades. Although this code has now been sidelined in favor of a newer version, it has continued to be improved up to the present time.

On a historical note, the first attempts to study primordial binaries in star clusters by direct  $N$ -body techniques (Aarseth 1975, 1980) were made at a time when few such systems were observed in open clusters and none in globular clusters (Trimble 1980). However, from recent obser-

vations of star-forming regions, it is now desirable to consider models with an arbitrary binary membership up to 100%, which can be handled by the new codes (Kroupa 1995a). It may also be recorded here that a comparison of three different methods—direct  $N$ -body, Monte Carlo, and the fluid-dynamical approach—did much to restore confidence in the validity of results (Aarseth, Hénon, & Wielen 1974), following earlier work on the exponential error growth in  $N$ -body systems (Miller 1964).

## 8. LARGE- $N$ CODES

With the rapidly increasing cost of studying larger systems, it is natural to seek other ways to speed up the calculation. Several attempts in this direction were made during my career; however, they all seemed to end up like a river running out in the desert, although much experience was gained.

An early formulation was based on the expansion of the potential in spherical harmonics, with application to the galactic tidal field (Aarseth 1967). A fourth-order expansion was employed, together with an interpolation for the effect inside a given shell. This technique anticipated tree codes which appeared much later. The idea was to study star clusters with well-developed core-halo structure, in which the core particles would be advanced by direct integration. Such a scheme was actually implemented successfully, but the results remained unpublished and further work was overtaken by events. The gain in treating the central particles which have small time steps is fairly substantial because only the effect of *external* shells needs to be taken into account instead of summation over a large number of distant particles.

Another approach resulted in a hybrid code based on gas dynamics, where again the core was treated by the direct method of NBODY5 (Aarseth & Bettwieser 1986). Unfortunately, this sophisticated development was not continued, although the gasdynamical description has recently been revived (Hemsendorf 1998) with a code based on the new NBODY6 (see § 10).

A considerable effort was also devoted to developing and testing a collisional tree code (NBODYX) which contained many new features (McMillan & Aarseth 1993). One new idea here was that the tree structure is not recalculated at each step but that the cells deform with time, using terms up to octupole if necessary. Instead of employing the low-order leapfrog integrator as is usual in tree codes, the more accurate force evaluation was combined with fourth-order differences and individual time steps for both the cells and particles. Moreover, hierarchical block steps facilitated a novel time step scheduling as well as vectorization. This development made use of point masses with standard KS regularization and was therefore fully collisional, making it a tool for star cluster simulations. Because of the slow

asymptotic approach to the predicted  $N \log N$  scaling, it was estimated that the tree code would become competitive with NBODY5 for  $N \geq 10^4$ . This contention has not been put to the test yet, but it was encouraging to see good agreement with NBODY5 for core collapse of a system containing 1024 particles. Now that faster workstations are becoming available, it would be desirable to revitalize this project for anyone without access to the HARP super-computer.

## 9. CHAIN REGULARIZATION

Given the existence *and* application of three-body regularization, it is remarkable that it took another 17 years before a fourth body was added (Mikkola & Aarseth 1990). In retrospect, the regularization of two-body forces along a *chain* of interparticle vectors might not appear a natural way to describe the motions in few-body systems. However, the main reason why such a formulation is so effective is that the other interactions are nondominant and the decision making is relatively inexpensive, although the programming complications may act as a deterrent to others. Below we summarize the main steps which have much in common with the three-body formulation of § 4, and the two methods are in fact equivalent for  $N = 3$ .

We begin by introducing relative momentum vectors  $W_k$  which are related recursively to the physical momenta  $p_k$  by

$$W_k = W_{k-1} - p_k; \quad (k = 2, \dots, N-2), \quad (23)$$

with  $W_1 = p_1$  and  $W_{N-1} = p_N$ . These momenta and corresponding relative coordinates  $R_k$  are substituted into the Hamiltonian, which becomes

$$H = \sum_{k=1}^{N-1} \frac{1}{2} \left( \frac{1}{m_k} + \frac{1}{m_{k+1}} \right) W_k^2 - \sum_{k=2}^N \frac{1}{m_k} W_{k-1} \cdot W_k - \sum_{k=1}^{N-1} \frac{m_k m_{k+1}}{R_k} - \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{R_{ij}}. \quad (24)$$

We now employ the time transformation  $dt = g d\tau$  where  $g = 1/L$  is the inverse Lagrangian, which has certain advantages over the product of the distances adopted before. The regularized Hamiltonian  $\Gamma^* = g(H - E)$  again gives rise to equations of motion in the form of equation (21). Inspection of these equations reveals that the two-body solutions are regular for all  $R_k \rightarrow 0$ . Finally, on the technical side, the relative coordinates and momenta are recovered from the standard KS relations by

$$\begin{aligned} R_k &= \mathcal{L}_k Q_k, \\ W_k &= \frac{\mathcal{L}_k P_k}{2Q_k^2}, \end{aligned} \quad (25)$$

where  $\mathcal{L}_k$  is the generalized Levi-Civita matrix equation (13).

An example of a chain structure is illustrated in Figure 1. Here the pairwise distances  $R_1$  and  $R_3$  connect two binaries separated by the vector  $R_2$ , which may have been larger initially. The solid lines denote regularized two-body interactions, whereas the dotted lines represent the nonchained interactions given by the last term of equation (24). Switching to a new dominant configuration would occur if either of the dotted distances became smaller than the neighboring chained distances.

A more elegant formulation was given subsequently with application to systems with arbitrary memberships (Mikkola & Aarseth 1993). Implementation in NBODY5 proved successful after the considerable effort of combining two different methods in order to advance all the motions consistently. External perturbations are included in an analogous way to KS, except that the tidal force on each dominant two-body term must now be obtained. A further significant improvement took place when the so-called slowdown procedure was introduced (Mikkola & Aarseth 1996). The idea here is to employ the principle of adiabatic invariance to scale small perturbations and the corresponding physical time step by a factor exceeding unity such that one orbit may represent several Kepler orbits. Slowing down the orbital velocity of a binary permits arbitrarily small periods to be studied; this replaces the previous use of the unperturbed two-body approximation. An example of such a system is a superhard binary entering an existing subsystem treated by chain regularization.

The chain algorithms are ideally suited to studying strong triple and binary-binary interactions in star cluster simulations. However, only compact subsystems are selected for treatment, the main reason being the high-order integrator (Bulirsch & Stoer 1966), which requires many function evaluations for each step, and hence it is desirable to limit the number of perturbers. The scheme is very flexible because the membership may vary (up to six at present) until the final configuration consists of one binary well separated from a single particle or another binary, which

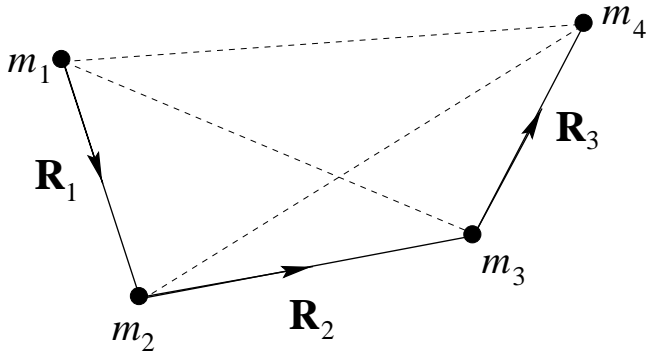


FIG. 1.— Chain configuration of four interacting particles

can then be treated by KS or direct integration. An attractive feature of the chain description is that much of the decision making employs the pairwise distances  $R_k$ , which are updated throughout the integration. In this way, the location of any binaries in the chain may readily be identified or candidates for escape removal determined. To indicate the degree of sophistication involved, a final system of five members may satisfy the condition of hierarchical stability when considered as a triple, provided two close binaries are present.

## 10. HERMITE SCHEME

Although somewhat cumbersome, the divided difference formulation proved its worth during some 30 years of use. However, Hermite integration, which takes its name from the Hermite interpolation polynomial, is now the preferred alternative as regards simplicity, accuracy, and performance. This method was originally developed for the special-purpose HARP computer (Makino 1991) but also turned out to be effective on workstations and parallel supercomputers.

The basic idea is very simple and relies on employing a Taylor series for both the force,  $F$ , as well as its time derivative,  $F^{(1)}$ . Consequently, we write an expansion up to third order in the force and its derivative as

$$\begin{aligned} F &= F_0 + F_0^{(1)}t + \frac{1}{2}F_0^{(2)}t^2 + \frac{1}{6}F_0^{(3)}t^3, \\ F^{(1)} &= F_0^{(1)} + F_0^{(2)}t + \frac{1}{2}F_0^{(3)}t^2, \end{aligned} \quad (26)$$

where  $t$  now represents the convergent time interval. Here  $F^{(1)}$  is obtained explicitly in terms of all the relative coordinates and velocities, which is roughly similar to the cost of a full force evaluation. Given the values of  $F$  and  $F^{(1)}$  at the beginning and end of a time step, the solutions for the two higher derivatives take the form

$$\begin{aligned} F_0^{(2)} &= [-3(F_0 - F) - (2F_0^{(1)} + F^{(1)})t] \frac{2}{t^2}, \\ F_0^{(3)} &= [2(F_0 - F) + (F_0^{(1)} + F^{(1)})t] \frac{6}{t^3}. \end{aligned} \quad (27)$$

These expressions give rise to the coordinate and velocity corrector for particle  $i$  with step  $\Delta t$ ,

$$\begin{aligned} \Delta r_i &= \frac{1}{24}F_0^{(2)}\Delta t^4 + \frac{1}{120}F_0^{(3)}\Delta t^5, \\ \Delta v_i &= \frac{1}{6}F_0^{(2)}\Delta t^3 + \frac{1}{24}F_0^{(3)}\Delta t^4. \end{aligned} \quad (28)$$

This fourth-order scheme has the advantage of being self-starting and may be used as it stands. However, significant gains can be achieved by introducing hierarchical time steps

by the simple rule

$$\Delta t_n = \Delta t_1 \left(\frac{1}{2}\right)^{n-1}, \quad (29)$$

where  $\Delta t_1$  is a scaling factor. Hence the time step is truncated to the nearest quantized value after employing an expression of the type equation (7). This facilitates the prediction of coordinates and velocities which can now be made just once for all particles needing to be advanced at a given time. For completeness, these predictions are given by

$$\begin{aligned} r_j &= \left[\left(\frac{1}{6}F^{(1)}\delta t'_j + \frac{1}{2}F\right)\delta t'_j + v_0\right]\delta t'_j + r_0, \\ \dot{r}_j &= \left(\frac{1}{2}F^{(1)}\delta t'_j + F\right)\delta t'_j + v_0, \end{aligned} \quad (30)$$

with  $\delta t'_j = t - t_j$  as before.

The advantage of reducing the number of predictions in a high-order scheme was not realized before and acted as a stumbling-block to progress. However, it is interesting to note an early effort with a low-order scheme based on quantized times associated with the force calculation as well as the use of explicit expressions for the derivative (Hayli 1974). We also note that a high-order explicit derivative scheme was already in use for small- $N$  planetary systems (Lecar, Loeser, & Cherniak 1974). Yet somehow the elegant Hermite integration method eluded discovery until much later and can be said to have been stimulated by hardware developments.

Hermite integration has also been adapted to the Ahmad-Cohen (1973) scheme (Makino & Aarseth 1992). This formulation has several advantages, the most important being the absence of large derivative corrections for neighbor changes which occur when using divided differences. It has given rise to the NBODY6 code, which should be considered a successor to NBODY5 and has now reached a satisfactory final state (Aarseth 1994, 1999a). Somewhat surprisingly, the new code is in fact slightly faster than NBODY5 on a standard workstation for  $N = 1000$ . Further work on adaptation to parallel supercomputers (T3E) has resulted in a new version, denoted by NBODY6<sup>++</sup> (Spurzem 1999).

In order to achieve a uniform treatment, NBODY6 has been converted to the Hermite KS formulation. This development entails differentiating equations (15) and (16), utilizing the properties  $\mathcal{L}'(\mathbf{u}) = \mathcal{L}(\mathbf{u}')$  and  $\mathbf{P}' = R\dot{\mathbf{P}}$ . Although the new equations of motion contain more terms and require velocity predictions of perturbers as well as a KS velocity prediction to evaluate  $\dot{\mathbf{P}}$ , the extra effort is compensated for by a simpler treatment. The slowdown principle (Mikkola & Aarseth 1996) was employed here for some time before it was generalized to chain regularization. Thus one period of a weakly perturbed binary ( $\gamma < 5 \times 10^{-5}$ ) may now represent several periods, as determined by a sophisticated scheme for decision making in which the slowdown

modification changes by a factor of 2 at the apocenter turning point, when a new polynomial initialization is carried out. If desired, there is no need to introduce unperturbed two-body motion when using the slowdown approach here because the intervals for checking could become arbitrarily large in terms of orbital periods. However, for technical reasons of decision making it is convenient to distinguish between the two different states of motion because an unperturbed binary can be treated as a single particle.

## 11. STUMPPFF KS REGULARIZATION

Although experience with the Hermite KS scheme has been favorable, the use of energy stabilization (Aarseth 1985) was criticized (Funato et al. 1996) because angular momentum is no longer conserved, and a time-symmetric method was recommended. This procedure was introduced to avoid a small systematic error in the semimajor axis for long-term integration, whereas its effect on the eccentricity is reduced by the slowdown algorithm for small perturbations. In any event, the time-symmetric method turned out to be unsuitable for general  $N$ -body implementations, but the idea of a Hermite iteration was subsequently employed in planetesimal simulations for the dominant solar interaction (Kokubo, Yoshinaga, & Makino 1998).

In turning to yet another method of two-body regularization after all these years, we aim to improve the solutions for small perturbations by exploiting the so-called Stumpff (1962) functions, which predate even KS itself. Here the idea is to modify the two last terms of the KS polynomial derivatives corresponding to equation (27) by including the contribution from the neglected terms in the expansion. With the notation  $\Omega = -\frac{1}{2}h$  and  $\mathbf{Q} = \mathcal{L}'\mathbf{P}$ , the basic equation of motion for the KS coordinates  $\mathbf{U}$  (previously denoted by  $\mathbf{u}$ ) takes the form

$$\mathbf{U}^{(2)} = -\Omega\mathbf{U} + \frac{1}{2}R\mathbf{Q}. \quad (31)$$

The combined Hermite equations of motion at the start of a step are then given by

$$\begin{aligned} \mathbf{U}_0^{(2)} &= -\Omega_0\mathbf{U}_0 + \mathbf{f}_0^{(2)}, \\ \mathbf{U}_0^{(3)} &= -\Omega_0\mathbf{U}_0 + \mathbf{f}_0^{(3)}, \end{aligned} \quad (32)$$

where  $\mathbf{f}_0^{(2)} = \frac{1}{2}R\mathbf{Q}$  is the perturbed force function evaluated after the previous predictor cycle and  $\mathbf{f}_0^{(3)}$  is its corresponding regularized derivative. Taylor series prediction gives  $\mathbf{U}$ ,  $\mathbf{U}'$  to order  $\mathbf{U}^{(5)}$ , where the two highest orders are also known initially from explicit derivatives to sufficient accuracy. This prediction employs pairwise Stumpff (1962) functions in the two last terms, where the modified coefficients

take the form

$$\tilde{c}_n(z) = n! \sum_{k=0}^{\infty} \frac{(-z)^k}{(n+2k)!} \quad (33)$$

with argument  $z$  related to the step  $\Delta\tau$  by  $z = -\frac{1}{2}h\Delta\tau^2$ . As usual, the regularized time step is given by a constant fraction  $\eta_u/(2\pi)$  of the period,

$$\Delta\tau = \eta_u \left( \frac{1}{2|h|} \right)^{1/2}, \quad (34)$$

together with an appropriate shortening due to the perturbation. The expansion equation (33) converges rapidly for small  $z$ , and it suffices to include 12 terms in the summation which must be repeated at every step.

Following prediction of coordinates and velocities to order  $\dot{F}$ , we obtain the perturbation at the end of the step. Corresponding new perturbative functions are given by

$$\begin{aligned} f^{(2)} &= (\Omega_0 - \Omega)U + \frac{1}{2} RQ, \\ f^{(3)} &= (\Omega_0 - \Omega)U' - \Omega'U + \frac{1}{2} R'Q + \frac{1}{2} RQ'. \end{aligned} \quad (35)$$

This enables construction of the higher derivatives by the Hermite rule equation (27), resulting in the expressions

$$\begin{aligned} U_0^{(4)} &= -\Omega_0 U_0^{(2)} + f_0^{(4)}, \\ U_0^{(5)} &= -\Omega_0 U_0^{(3)} + f_0^{(5)}. \end{aligned} \quad (36)$$

The provisional solution, which also involves the modified c-functions, is improved by one iteration without reevaluating the perturbations, whereas the energy integration

$$\Omega' = -U' \cdot Q \quad (37)$$

remains the same as for standard Hermite integration.

The new formulation also admits the slowdown procedure (Mikkola & Aarseth 1996) which scales the perturbation terms and time derivative  $t' = U \cdot U$  by the appropriate factor. Although more operations are now required for each step, a satisfactory accuracy can be obtained with about half the number of steps, and the cost may actually be less but depends to some extent on the number of perturbers. Finally, we note that the new scheme may also be used without the modified Stumpff functions. However, in that case the errors are significantly larger for both weakly and strongly perturbed binaries.

## 12. HIERARCHICAL SYSTEMS

Recent  $N$ -body simulations with primordial binaries show that hierarchies play an important role in the cluster evolution (Mardling & Aarseth 1999b). Such systems tend

to be long-lived, and direct integration can be prohibitive because of the short periods involved. For some time, this problem was circumvented by adopting unperturbed motion for the inner binary based on various stability criteria obtained numerically (Harrington 1977; Eggleton & Kiseleva 1995).

A new approach, based on the correspondence with the chaos boundary in the binary-tides problem (Mardling 1995), has resulted in a semianalytical stability criterion which holds for a wide range of mass ratios and arbitrary outer eccentricities (see Mardling & Aarseth 1999a). The critical outer pericenter distance is given in terms of the inner semimajor axis,  $a_{\text{in}}$ , by

$$R_p^{\text{out}} = C \left[ (1 + q_{\text{out}}) \frac{(1 + e_{\text{out}})}{(1 - e_{\text{out}})^{1/2}} \right]^{2/5} a_{\text{in}}, \quad (38)$$

where  $q_{\text{out}} = m_3/(m_1 + m_2)$  is the outer mass ratio,  $e_{\text{out}}$  is the corresponding eccentricity, and  $C \simeq 2.8$  is determined empirically. This criterion applies to coplanar prograde orbits, which is the most unstable case, and ignores a weak dependence on the inner eccentricity and mass ratio  $m_1/m_2$ . However, numerical tests suggest a modest inclination correction factor of up to 30%, in accord with early experiments (Harrington 1972).

A triple with outer pericenter satisfying  $a_{\text{out}}(1 - e_{\text{out}}) > R_p^{\text{out}}$  is defined to be stable, in which the inner binary experiences no secular effects and we neglect the short-term fluctuations. Such systems are converted to one KS solution, using the center-of-mass approximation for  $m_1 + m_2$  such that the period is now given by the *outer* component, thereby increasing the orbital timescale considerably. Quadruples and higher-order binary systems of the type  $[B_i - B_j]$  or  $[[B_i - B_j] - B_k]$  may also be treated in a similar manner by including appropriate correction factors to the basic relation (eq. [38]), as well as to the energy budget. The new stability criterion has been tested extensively and has turned out to be very effective, particularly because the boundary is more stringent than the numerically derived fitting functions.

The above development is concerned with the question of escape of the outer component. However, the alternative outcome of exchange with one of the inner components is also relevant. The latter process may be examined by the Zare (1977) criterion:

$$(c^2 E)_{\text{crit}} = -\frac{G^2 f^2(\rho) g(\rho)}{2(m_1 + m_2 + m_3)}, \quad (39)$$

where  $c$  is the angular momentum and  $f(\rho)$  and  $g(\rho)$  are given in terms of the masses. Thus if  $c^2 E < (c^2 E)_{\text{crit}}$  (with  $E < 0$ ), there can be no exchange and the inner binary identity is preserved. However, exchange does not necessarily occur if the criterion is violated but if it does, escape invari-

ably takes place. Moreover, we note that for zero inclination, the escape boundary lies above the exchange limit for  $q_{\text{out}} < 5$ , which is usually the case in star cluster simulations.

An attempt has been made to model the cyclic variation of the inner binary (Kozai 1962) due to the outer component (Mardling & Aarseth 1999b). This is achieved by a double averaging procedure (Mardling & Eggleton 1999), where only systems with a high maximum eccentricity are considered. In addition, contributions to the change in eccentricity from oblateness terms, tidal interactions, as well as general relativistic precession are included, but not yet stellar spin for technical reasons. This treatment may lead to the onset of tidal circularization, with subsequent orbital shrinkage inside hierarchical systems.

On the dynamical side, it is now becoming clear why hierarchies are so important in star cluster dynamics. The basic process of formation was already noted in early binary-binary scattering experiments (Mikkola 1983) and is essentially one of exchange. In some sense, the formation of a stable triple may be viewed as a new binary. However, this usually involves two strongly interacting binaries and is therefore a *two-body* process, whereas the classical binary formation mechanism requires at least three bodies inside a small volume. The promotion of mass segregation through mass loss from the dominant single stars leads to an increased binary population in the core, which can approach 50% in the late stages even when starting with values as low as 5%. Although the formation rate turns out to be low, many of the binaries formed are fairly hard and cannot be broken up by single stars. This may be contrasted with new binaries originating at a slower rate by the classical mechanism, and even being hard does not ensure survival. In the subsequent evolution to a superhard state, such binaries are subject to occasional encounters with existing primordial binaries and are therefore likely to be disrupted. This general picture agrees qualitatively with the actual outcome in which new binaries are relatively rare, whereas the number of stable hierarchies tends to increase with time. Note that here a primordial binary exchanging one or both components is still considered as primordial in view of its energy budget.

In the postcollapse phase of cluster evolution, the particle density in the core of even rich open clusters decreases significantly and allows high-order subsystems to be formed. Considering that the innermost binary may have a period of a few days or less, there is ample *hierarchical space* for such configurations to exist even allowing for relatively high eccentricities of the outer orbits. Not much is known about the mode of formation, even of quadruples, but it is likely to be connected with dynamical friction or large density fluctuations. The treatment of such systems is sufficiently time-consuming that it has been necessary to develop algorithms for up to six members to be represented as just one KS solution. It also follows that multiple hierarchies tend to be

massive and have large cross sections which promote dynamical activity.

### 13. THE HARP EXPERIENCE

The availability of the HARP special-purpose computer (Makino, Kokubo, & Taiji 1993) has created unique opportunities for star cluster simulations. The challenge of having a one-user supercomputer in one's office since 1994 has also inspired an effort toward greater realism. Having overcome the technical problems relatively painlessly (Aarseth 1996b), the main work has been concerned with developing astrophysical algorithms, some of which will be highlighted below.

In the first instance, the subject of primordial binaries has been placed on a more observational footing. Given the initial binary fraction  $f_b = N_b/(N_b + N_s)$ , with  $N_s$  the number of single stars, we need to specify the IMF as well as the period distribution. The main choice of IMF is based on stars in the solar neighborhood (Kroupa, Tout, & Gilmore 1993), with typical mass limits of  $(15-0.2)m_\odot$ . For simplicity, we adopt a flat distribution in  $\log a$  subject to a lower cutoff of 4 days, with component masses sampled at random from the IMF or sometimes slightly correlated. However, a more realistic period distribution is also available (Kroupa 1995a). Considerations of efficiency on HARP due to the perturber treatment play a role in deciding the upper limit, which is usually chosen conservatively at about half the hard binary limit  $a_h = 120$  AU for a cluster with half-mass radius  $r_h = 3$  pc. Until recently, a modest binary fraction of  $f_b = 0.05$  was used for typical models with  $N_s = 10^4$  in order to demonstrate that this suffices to achieve significant effects. Note that nearly all the binaries in the adopted distribution are unperturbed (i.e.,  $\gamma < 10^{-6}$ ) at a given time, which speeds up the calculation.

Many stellar radii and masses evolve significantly over the long timescales associated with cluster simulations, where the lifetime may exceed 5 Gyr. In order to describe the relevant changes, we employ synthetic stellar evolution based on fast lookup functions (Tout et al. 1997). This scheme employs an integer index  $K$  for each star which defines its evolutionary type, ranging from a low-mass main-sequence star to the appropriate final state of white dwarf, neutron star, or black hole. Both the radius and core mass are available as functions of the time, where the lookup interval is sufficiently small to ensure a smooth change. In addition, we model stellar mass loss due to winds according to

$$\dot{m} = 4 \times 10^{-13} \frac{r^* L}{m} \quad (40)$$

in  $m_\odot \text{ yr}^{-1}$ , with a suitable average over radius and luminosity in solar units. Depending on the initial mass, the final

transition to degenerate object may involve substantial mass loss. For neutron star or black hole formation, we assign a velocity kick sampled from a Maxwellian which usually ensures escape from the cluster, although some massive binaries may be ejected without disruption. So far only solar metallicity has been employed in the fitting functions, but a new synthetic stellar evolution scheme is now available for a wide range of the metal abundance (Hurley & Tout 1999) and has been incorporated into NBODY6.

A considerable effort has been devoted to modeling chaotic motion in binaries as well as tidal circularization and even capture (Mardling & Aarseth 1999b). The former type is usually initiated after strong interactions which may yield a large (i.e.,  $e > 0.99$ ) final eccentricity. The subsequent evolution produces large tides on the stars and the eccentricity changes irregularly until the chaos boundary is crossed, whereupon normal circularization takes over. Binaries are selected for special treatment if either of the conditions  $a(1 - e) < 4r^*$  or  $t_{\text{circ}} < 2 \times 10^9$  yr are satisfied, where  $t_{\text{circ}}$  is the circularization time which depends on stellar type via a polytropic representation. The orbital shrinkage is considerable and provides a mechanism for producing binaries of short period. Because most of the relevant binaries are unperturbed during this phase, the additional cost is almost negligible.

The completion of tidal circularization sets the stage for Roche lobe mass transfer. Except for some binaries experiencing rapid stellar growth, the Roche condition  $r^* > r_{\text{RL}}$  is usually reached a few million years after orbit circularization. Mass transfer is implemented by a rapid binary evolution algorithm (Tout et al. 1997), which includes several important processes. Relevant corrections are made to the orbit as well as the overall energy budget to maintain conservation. The modeling of common envelope evolution leads to coalescence or even doubly degenerate binaries with periods of a few hours.

Binaries that have not yet reached the Roche lobe overflow condition may be subject to magnetic braking or gravitational radiation. The resulting orbital shrinkage is usually significant for separations of a few solar radii, with the corresponding rates  $\dot{a}_{\text{MB}} \propto a^{-4}$  and  $\dot{a}_{\text{GR}} \propto a^{-3}$  applicable to different stellar types. Depending on the mass ratio, this evolution may terminate in physical collision, with the possible formation of a blue straggler for main-sequence components. General stellar collisions also occur, and on rare occasions the relative motion may be hyperbolic and even lead to tidal capture. At present we use the pericenter collision criterion derived for main-sequence stars, given by

$$a(1 - e) < 1.7 \left( \frac{m_1 + m_2}{2m_1} \right)^{1/3} r_1, \quad (41)$$

where  $r_1$  is the largest stellar radius. Hopefully, a more appropriate criterion will be available for giants before too

long. The final outcome is predicted by a collision matrix which depends on stellar types and may give rise to observable exotic objects.

Following considerable code refinements since the earliest attempts of dealing with binaries (Aarseth 1996a), it is a pleasant surprise to see how well those results stand up with all the main effects included. Thus the presence of a few blue stragglers for a cluster with  $N = 5000$  is entirely consistent with recent results for larger systems. However, it is also fair to say that much experience has been gained during these formative years, and the whole project has matured to a stage which may encourage fruitful confrontation with observations. It also remains to be seen whether the present approach, with its heavy reliance on regularization, will prove to be more useful than a treatment which handles close encounters and compact subsystems by a local hierarchical tree structure (Portegies Zwart et al. 1998).

Looking to the future, a new and much more powerful GRAPE-6 (or HARP-6) is on schedule for year 2000, and the first prototype board has already been tested successfully (J. Makino 1999, private communication). Hence the prospects are good for an attack on the globular cluster problem. This challenge has resulted in new and improved algorithms for advancing the center of mass of perturbed binaries on HARP.<sup>3</sup> Thus the cumbersome procedure of setting a zero mask for all perturber masses before the force evaluation and replacing the correct values after every such step has been abandoned. In the old scheme the missing terms were formed directly on the host, but this led to loss of efficiency due to data communication and identification of joint perturbers when more than one center-of-mass particle is advanced at the same time. Instead, the total force and first derivative are now obtained on HARP, followed by a differential correction procedure on the host. Although not numerically elegant because the subtracted terms are evaluated with different precision on HARP, this procedure is sufficiently accurate for the present purpose and no undesirable effects have been seen for quite extreme situations, such as the termination of a compact triple in chain regularization. Even so, the treatment of many perturbed binaries places a considerable strain on the host machine, which is likely to be the bottleneck with the present integration scheme because the number of KS steps is in any case quite large. This raises further questions of code design and the development of new algorithms in order to push the limits of star cluster simulations to even larger  $N$ .

It is appropriate to end this section on modern  $N$ -body developments by listing my main technical collaborators and their contributions:

1. Seppo Mikkola: two-body and chain regularization,

<sup>3</sup> S. Aarseth 1999, GRAPE Users Community (Tokyo) at <http://grape.c.u-tokyo.ac.jp/grape>.

2. Rosemary Mardling: tidal evolution and hierarchical stability,
3. Chris Tout: stellar evolution and Roche mass transfer,
4. Douglas Heggie: tidal shocks and large- $N$  scaling (Aarseth & Heggie 1998).

#### 14. CONCLUSIONS

We have seen how an endeavor which started at almost the beginning of the computer revolution has involved a wide range of dynamical  $N$ -body problems. The general philosophy has been to develop algorithms which match the applications, and my main involvement has therefore turned out to be associated with code design. As is well known, this type of work requires much patience. However, in order to construct efficient codes, it is imperative to stay close to the dynamical problem, which is an ideal way to progress on two fronts.

The inspiration to improve codes comes from the availability of new algorithms, and in this respect luck has played a large part. Thus one might ask what would have happened if the elegant KS regularization method had not been invented in time to become such a vital element in the large  $N$ -body codes described here. Likewise, the whole multiple regularization edifice would undoubtedly be in limbo. The interested reader is referred to an article by Mikkola (1997), which contains some historical remarks on this subject, as well as a nice review. Hence one's achievements are inextricably linked to the whole scientific advancement which presents opportunities to be exploited. One problem with modern codes is that work never ends; there are so many things that can be done. When coupled with the need for testing which can become an all-consuming passion, it is hardly surprising that one can devote a lifetime to such activities. Finally, the entire effort of developing direct integration methods has been aimed at attacking the globular cluster problem as the K2 to be scaled the hard way, and this has involved a certain single-mindedness of purpose.

After these philosophical remarks it may be in order to list the main types of code which have been developed. These can be divided into separate categories as follows:

1. general-purpose dynamics with softening,
2. star cluster simulations,
3. cosmological modeling,
4. planetary formation,
5. few-body scattering experiments.

All these codes are freely available to the astronomical community on the World Wide Web.<sup>4</sup> However, some are rather large (30,000 lines and over 200 subroutines for NBODY6) with no external guidance except for the papers, and potential users would therefore benefit from personal consultation until the overdue book project has been completed. Nevertheless, all the direct  $N$ -body codes have been constructed on similar principles and contain sufficient internal documentation to follow the algorithms for those familiar with the papers. It will be clear from the topics covered in this Review that the massive effort has involved a number of collaborators, on both the astrophysical and technical side. Finally, in order to justify all the investments and notwithstanding progress, these  $N$ -body codes will hopefully continue to serve a useful purpose for some time to come.

I would like to express my appreciation of financial support from the Institute of Astronomy (formerly Institute of Theoretical Astronomy) since its foundation in 1967, as well as the relevant governmental funding bodies. Its many directors provided me with a privileged working environment which enabled this dedicated effort to be undertaken. It is also a great pleasure to thank all my collaborators and other colleagues who inspired my work and made it so enjoyable.

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<sup>4</sup> <http://www.ast.cam.ac.uk/~sverre>.

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